RINGS OVER WHICH EVERY MODULE IS PURELY (QUASI-)CONTINUOUS MODULES

RASOUL MORADI, YASER TOLOOEI AND NAJMEH DEHGHANI

ABSTRACT. We study four purity-based generalizations of the classical conditions C_2 and C_3 for modules, We defined the following notions on M_R:

- *Purely* C_2 (*or* $P.C_2$): if a submodule A of M_R is isomorphic to a pure submodule of M_R, then A is also a pure submodule of M_R.
- *Generalized purely* C_2 (*or* $G.P.C_2$): if a submodule A of M_R is isomorphic to a direct summand of M_R, then A is a pure submodule of M_R. (These modules were called purely C_2 in [1]).
- *Purely* C_3 (*or* $P.C_3$): if A_1 and A_2 are pure submodules of M_R with $A_1 \cap A_2 = 0$, then $A_1 \oplus A_2$ is a pure submodule of M_R.
- *Generalized purely* C_3 (*or* $G.P.C_3$): if A_1 and A_2 are direct summands of M_R with $A_1 \cap A_2 = 0$, then $A_1 \oplus A_2$ is a pure submodule of M_R.
- We call purely extending modules, in the sense of J. Clark, $P.C_1$ *modules*.

The module M_R is called *purely continuous* (*or p.continuous just for short*) if M_R has both $P.C_1$ and $P.C_2$, and *purely quasi-continuous* (*or p.quasi-continuous*) if it has $P.C_1$ and $P.C_3$. Moreover, we call M_R *generalized purely continuous* (*or g.p.continuous*) if it has $P.C_1$ and $G.P.C_2$ and it is called *generalized purely quasi-continuous* (*or g.p.quasi-continuous*) whenever it has $P.C_1$ and $G.P.C_3$. Our main result characterizes von Neumann regular rings as precisely those over which every (finitely generated) module satisfies any of these properties. In contrast, we show that rings over which every (finitely generated) module has the C_2 or C_3 property are exactly the semisimple Artinian rings. We also characterize rings whose finitely cogenerated modules are C_2 or C_3, and we investigate when projective modules satisfy these purity conditions. Several illustrative examples are provided.

1. MAIN RESULTS

First we present terminologies we used in this section. Let M be an R-module. The notations $N \leq M_R$, $N \leq_{ess} M_R$ and $N \leq_{\oplus} M_R$, mean that N is a submodule, essential submodule and a direct summand of M_R, respectively. $\mathrm{Soc}(M)$ denote the socle of M_R. We begin with the following theorem.

Theorem 1.1. *Let R be a ring. The following statements are equivalent:*

(a_1) *every R-module has* $P.C_2$ *property;*
(a_2) *every finitely generated R-module has* $P.C_2$ *property;*
(b_1) *every R-module has* $G.P.C_2$ *property;*
(b_2) *every finitely generated R-module has* $G.P.C_2$ *property;*
(c_1) *every R-module has* $P.C_3$ *property;*
(c_2) *every finitely generated R-module has* $P.C_3$ *property;*
(d_1) *every R-module has* $G.P.C_3$ *property;*
(d_2) *every finitely generated R-module has* $G.P.C_3$ *property;*

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(e_1) *R is a von Neumann regular ring.*

Corollary 1.2. *The following statements are equivalent for a ring R:*

(a) *every (finitely generated) R-module is p.continuous;*
(b) *every (finitely generated) R-module is g.p.continuous;*
(c) *every (finitely generated) R-module is p.quasi-continuous;*
(d) *every (finitely generated) R-module is g.p.quasi-continuous;*
(e) *R is a von Neumann regular ring.*

Remark 1.3. Regarding Theorem 1.1, it is interesting to characterize rings over which every (finitely generated) module has the C_2 (C_3) property. In Proposition 1.7, we prove that rings over which every module has C_2 (C_3) property, are precisely semisimple Artinian rings. On the other hand, since C_3 implies G.P.C_3, if every finitely generated R-module has C_3 property, then R should be a von Neumann regular ring. The following example is showing that the converse does not hold in general, even in the commutative case.

Example 1.4. This example illustrates that there exists a von Neumann regular ring R and a finitely generated R-module M such that M_R has no C_3 property. To see this, let F be a field and each $F_n = F$ for $n = 1, 2, 3, \cdots$. Assume that R is the F-subalgebra of $\prod F_n$ generated by 1 and $\oplus_{n\geq 1} F_n$. By [2, Example 1.8], R is a commutative von Neumann regular ring. Now let $x \in \prod F_n \setminus R$, $N = R + xR$ and $M = R \oplus N$. It is clear that M_R is finitely generated. We prove that M_R has not C_3 property. On the contrary, suppose that M_R has C_3 property. Let $A = R \oplus 0$ and $B =< (r, r) \mid r \in R >$. It is easy to check that A and B are direct summands of M_R such that $A \cap B = 0$. By C_3 property on M_R, $A \oplus B = R \oplus R \leq_{\oplus} M_R$. Therefore $R \leq_{\oplus} N_R$ and so $R \oplus T = N$ for some $T \leq N_R$. Thus there exist $t \in T$ and $r \in R$ such that $x = r + t$. Let for each $n \geq 1$, $e_n = (0, 0, \cdots, 1, 0, 0, \cdots)$. Thus $te_n = xe_n - re_n \in T \cap R = 0$ for all $n \geq 1$ and so $t = 0$. Therefore $x = r \in R$, a contradiction.

Theorem 1.1 motivated us to ask ourselves "whether rings over which every finitely generated projective module has P.C_i (G.P.C_i) $2 \leq i \leq 3$, property are von Neumann regular or not". In the next, we give an example which is showing that the answer of this question is negative. While in Proposition 1.6, we give a necessary and sufficient condition under which the answer of our question is positive. First we recall that a ring R is said to be *quasi-Frobenius* if R is a right Noetherian right self injective ring.

Example 1.5. Let R be a quasi-Frobenius ring. It is routine to see that R is semihereditary if and only if R is von Neumann regular. Moreover, it is well known that quasi-Frobenius rings are precisely the ones over which a projective module is injective and vice versa [3, Theorem 15.9]. Therefore every finitely generated projective R-module is injective and so has C_2 property. While R is not necessarily a von Neumann regular ring.

Proposition 1.6. *Let R be a ring. Then R is (right) semihereditary and every finitely generated projective R-modules has* C_2 (P.C_2, G.P.C_2, C_3, P.C_3, G.P.C_3) *if and only if R is a von Neumann regular ring.*

Following the above question and Theorem 1.1, we are curious about rings over which every (finitely generated) module has C_2 or C_3 property. We recall that an R-module M is said to be d-*Rickart* whenever for every R-endomorphism $f : M \to M$,

$\mathrm{Im} f \leq_{\oplus} M_R$. It was proved that every (free) R-module is d-Rickart if and only if R is a semisimple Artinian ring [4, Theorem 2.24].

Proposition 1.7. *Let R be a ring. The following statements are equivalent:*

(a) *every (finitely generated) R-module has* C_2 *property;*
(b) *every (finitely generated) R-module has* C_3 *property;*
(c) *every (finitely generated) R-module is* d-*Rickart;*
(d) *R is a semisimple Artinian ring.*

Recall that a ring R is said to be *a right* V*-ring* if every simple R-module is injective. Besides, an R-module M is called *finitely cogenerated* whenever $\mathrm{Soc}(M)$ is finitely generated and $\mathrm{Soc}(M) \leq_{ess} M_R$. In next proposition, we prove the dual of Proposition 1.7.

Proposition 1.8. *Let R be a ring. The following statements are equivalent:*

(a) *every finitely cogenerated R-module has* C_2 *property;*
(b) *every finitely cogenerated R-module has* C_3 *property;*
(c) *R is a right* V*-ring.*

Proposition 1.9. *Let R be a ring. The following statements are equivalent:*

(a) *every (resp., finitely generated) free R-module has* $\mathrm{G.P.C}_3$ *property;*
(b) *every (resp., finitely generated) projective R-module has* $\mathrm{G.P.C}_3$ *property;*
(c) *every (resp., finitely generated) free R-module has* $\mathrm{G.P.C}_2$ *property;*
(d) *every (resp., finitely generated) projective R-module has* $\mathrm{G.P.C}_2$ *property.*

We end this paper with the following characterization of domains over which every free (projective) module has $\mathrm{P.C}_i$ (C_i, $\mathrm{G.P.C}_i$) property.

Proposition 1.10. *Let R be a domain. The following statements are equivalent:*

(a_1) *every projective R-module has* $\mathrm{P.C}_3$ (C_3) *property;*
(a_2) *every projective R-module has* $\mathrm{G.P.C}_3$ *property;*
(a_3) *every free R-module has* $\mathrm{P.C}_3$ (C_3) *property;*
(a_4) *every free R-module has* $\mathrm{G.P.C}_3$ *property;*
(a_5) R^2_R *has* $\mathrm{P.C}_3$ (C_3) *property;*
(a_6) R^2_R *has* $\mathrm{G.P.C}_3$ *property;*
(b_1) *R is a division ring;*
(c_1) *every projective R-module has* $\mathrm{P.C}_2$ (C_2) *property;*
(c_2) *every projective R-module has* $\mathrm{G.P.C}_2$ *property;*
(c_3) *every free R-module has* $\mathrm{P.C}_2$ (C_2) *property;*
(c_4) *every free R-module has* $\mathrm{G.P.C}_2$ *property;*
(c_5) R^2_R *has* $\mathrm{P.C}_2$ (C_2) *property;*
(c_6) R^2_R *has* $\mathrm{G.P.C}_2$ *property.*

Remark 1.11. We note that the condition "domain" in Proposition 1.10, is necessary. Because, following Theorem 1.1, if R is a von Neumann regular ring, then every (free) R-module has C_3 ($\mathrm{P.C}_3$ and $\mathrm{G.P.C}_3$) property. In addition, over a right Noetherian right self injective (quasi-Frobenius) ring R, every free R-module is injective and so has C_3 ($\mathrm{P.C}_3$ and $\mathrm{G.P.C}_3$) property. While these kind of rings are not necessarily division ring.

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Rasoul Moradi, Department of Mathematical Sciences, Razi University, Kermanshah, Iran, 67149-67346.

e-mail: rasol.moradi@razi.ac.ir

Yaser Tolooei, Department of Mathematical Sciences, Razi University, Kermanshah, Iran, 67149-67346.

e-mail: y.toloei@razi.ac.ir

Najmeh Dehghani, Faculty of Intelligent Systems Engineering and Data Science, Persian Gulf University, Bushehr, Iran 751-6913817.

e-mail: n.dehghani@pgu.ac.ir